Mathematical Induction

Guess and Prove Formulas by Induction

(All Mathematical Induction parts are not shown.)

1. **(i)** $1 + (1 + 9) + (1 + 9 + 25) + \dots + [1^2 + 3^2 + 5^2 + \dots + (2n-1)^2]$

$$= \sum_{i=1}^{n} \left[1^2+3^2+5^2+\dots+(2i-1)^2\right] = \sum_{i=1}^{n}\sum_{k=1}^{i}(2k-1)^2 = \sum_{i=1}^{n}\sum_{k=1}^{i}\left(4k^2-4k+1\right)$$

$$= \sum_{i=1}^{n}\left[4\sum_{k=1}^{i}k^2 - 4\sum_{k=1}^{i}k + \sum_{k=1}^{i}1\right] = \sum_{i=1}^{n}\left[4\times\frac{1}{6}i(i+1)(2i+1) - 4\times\frac{1}{2}i(i+1)+i\right] = \sum_{i=1}^{n}\frac{1}{3}i\left(4i^2-1\right)$$

$$= \frac{1}{3}\left[\sum_{i=1}^{n}4i^3 - \sum_{i=1}^{n}i\right] = \frac{1}{3}\left[4\times\frac{n^2(n+1)^2}{4} - \frac{n(n+1)}{2}\right] = \frac{1}{6}\left[2n^2(n+1)^2 - n(n+1)\right] = \frac{1}{6}n(n+1)\left(2n^2+2n-1\right)$$

(ii) $a + 3(a+b) + 6(a+2b) + \dots + \frac{n}{2}(n+1)[a+(n-1)b]$

$$= a\left[1+3+6+\dots+\frac{n(n+1)}{2}\right] + b\left[3+6\times 2+\dots+\frac{(n-1)n(n+1)}{2}\right] = \frac{a}{2}\sum_{i=1}^{n}i(i+1) + \frac{b}{2}\sum_{i=2}^{n}(i-1)i(i+1)$$

$$= \frac{a}{6}\sum_{i=1}^{n}[i(i+1)(i+2)-(i-1)i(i+1)] + \frac{b}{8}\sum_{i=2}^{n}[(i-1)i(i+1)(i+2)-(i-2)(i-1)i(i+1)]$$

$$= \frac{a}{6}[n(n+1)(n+2)-(1-1)1(1+1)] + \frac{b}{8}[(n-1)n(n+1)(n+2)-(2-2)(2-1)2(2-1)]$$

$$= \frac{n(n+1)(n+2)}{24}[4a+3(n-1)b]$$

2. $$2\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\dots\left(1-\frac{1}{n^2}\right) = 2\prod_{i=2}^{a}\left(1-\frac{1}{i^2}\right) = 2\prod_{i=2}^{a}\frac{i^2-1}{i^2} = 2\prod_{i=2}^{a}\frac{(i+1)(i-1)}{i^2} = 2\prod_{i=2}^{a}\frac{i+1}{i}\prod_{i=2}^{a}\frac{i-1}{i}$$

$$= 2\left[\frac{a+1}{2}\right]\left[\frac{1}{a}\right] = \frac{a+1}{a}$$

3. $$\frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x} - \frac{\sin\left(n-\frac{1}{2}\right)x}{2\sin\frac{1}{2}x} = \frac{\sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x}{2\sin\frac{1}{2}x} = \frac{2\cos nx\sin\frac{1}{2}x}{2\sin\frac{1}{2}x} = \cos nx$$

$$\sum_{i=1}^{n}\cos ix = \sum_{i=1}^{n}\left[\frac{\sin\left(i+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x} - \frac{\sin\left(i-\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}\right] = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x} - \frac{\sin\left(1-\frac{1}{2}\right)x}{2\sin\frac{1}{2}x} = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x} - \frac{1}{2}$$

$$\therefore \quad \frac{1}{2} + \cos x + \cos 2x + \dots + \cos nx = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}$$

4. Let $u_r = \dfrac{a(a-1)\dots(a-r+1)}{b(b-1)\dots(b-r+1)}$, $\quad v_r = \dfrac{a(a-1)\dots(a-r+1)}{b(b-1)\dots(b-r+2)}$ $\quad r > 1$ for v_r

$$v_{r+1}-v_r=\frac{a(a-1)\dots(a-r)}{b(b-1)\dots(b-r+1)}-\frac{a(a-1)\dots(a-r+1)}{b(b-1)\dots(b-r+2)}=\frac{a(a-1)\dots(a-r+1)}{b(b-1)\dots(b-r+1)}[(a-r)-(b-r+1)]$$

$\therefore$ $v_{r+1} - v_r = (a - b - 1) \quad u_r \quad \Rightarrow \quad \sum_{i=2}^{n}[v_{r+1}-v_r]=(a-b-1)\sum_{i=2}^{n} u_r$

$$\therefore \quad \sum_{i=1}^{n} u_r = u_1+\frac{1}{a-b-1}[v_{n+1}-v_2]=\frac{a}{b}+\frac{1}{a-b-1}\left[\frac{a(a-1)\dots(a-n+1)}{b(b-1)\dots(b-n+2)}-\frac{a(a-1)}{b}\right]$$

5. $(1 - x)(1 - x^2)\dots(1- x^n) + x(1 - x^2)(1 - x^3)\dots(1- x^n) + x^2 (1 - x^3)\dots(1- x^n) +\dots + x^k (1 - x^{k+1})\dots(1 - x^n) +$
$\dots+ x^{n-1}(1 - x^n) + x^n \quad = 1$

6. $a_{n+1}=\frac{a_n+b_n}{2}$ (1) , $b_{n+1}=\frac{2a_nb_n}{a_n+b_n}$ (2)

(1) × (2), $a_{n+1}b_{n+1} = a_nb_n = a_{n-1}b_{n-1} = \dots = a_0b_0$

But $$\frac{\sqrt{a_n}-\sqrt{b_n}}{\sqrt{a_n}+\sqrt{b_n}}=\frac{a_n-\sqrt{a_nb_n}}{a_n+\sqrt{a_nb_n}}=\frac{a_n-\sqrt{a_{n-1}b_{n-1}}}{a_n+\sqrt{a_{n-1}b_{n-1}}}=\frac{\frac{a_{n-1}+b_{n-1}}{2}-\sqrt{a_{n-1}b_{n-1}}}{\frac{a_{n-1}+b_{n-1}}{2}+\sqrt{a_{n-1}b_{n-1}}}=\left(\frac{\sqrt{a_{n-1}}-\sqrt{b_{n-1}}}{\sqrt{a_{n-1}}+\sqrt{b_{n-1}}}\right)^2$$

Put $u_n=\frac{\sqrt{a_n}-\sqrt{b_n}}{\sqrt{a_n}+\sqrt{b_n}}$, then $u_n={u_{n-1}}^2={u_{n-2}}^{2^2}=\dots={u_0}^{2^n}$ …. (3)

But $u_n=\frac{\sqrt{a_n}-\sqrt{b_n}}{\sqrt{a_n}+\sqrt{b_n}}=\frac{a_n-\sqrt{a_nb_n}}{a_n+\sqrt{a_nb_n}}=\frac{a_n-\sqrt{a_0b_0}}{a_n+\sqrt{a_0b_0}}$, $u_0=\frac{a_0-\sqrt{a_0b_0}}{a_0+\sqrt{a_0b_0}}$ …. (4)

(4)↓(3), $\frac{a_n-\sqrt{a_0b_0}}{a_n+\sqrt{a_0b_0}}=\left(\frac{a_0-\sqrt{a_0b_0}}{a_0+\sqrt{a_0b_0}}\right)^{2^n}$. Solving, we have

$$a_n=\sqrt{a_0b_0}\left[\frac{\left(a_0+\sqrt{a_0b_0}\right)^{2^n}+\left(a_0-\sqrt{a_0b_0}\right)^{2^n}}{\left(a_0+\sqrt{a_0b_0}\right)^{2^n}-\left(a_0-\sqrt{a_0b_0}\right)^{2^n}}\right] \quad \text{and} \quad b_n=\frac{a_0b_0}{a_n}=\sqrt{a_0b_0}\left[\frac{\left(a_0+\sqrt{a_0b_0}\right)^{2^n}-\left(a_0-\sqrt{a_0b_0}\right)^{2^n}}{\left(a_0+\sqrt{a_0b_0}\right)^{2^n}+\left(a_0-\sqrt{a_0b_0}\right)^{2^n}}\right]$$

7. $x_n=\frac{\alpha x_{n-1}+\beta}{\gamma x_{n-1}+\delta}$ …. (1)

Consider two variables y_n and z_n such that $y_n = \alpha\, y_{n-1} + \beta\, z_{n-1}$ …. (2) $z_n = \gamma\, y_{n-1} + \delta z_{n-1}$ …. (3)

We put $x_n=\frac{y_n}{z_n}$, then we can get (1) .

Now, (2) + λ(3), $y_n + \lambda\, z_n = (\alpha +\lambda\gamma)\, y_{n-1} + (\beta + \lambda\delta)\, z_{n-1}$ …. (4)

Let us choose λ such that $\beta + \lambda\delta = \lambda(\alpha +\lambda\gamma)$ or $\gamma\lambda^2 + (\alpha - \delta)\, \lambda - \beta = 0$ …. (5)

$\therefore \quad \lambda_1=\frac{-(\alpha-\delta)+\sqrt{(\alpha-\delta)^2+4\gamma\beta}}{2\gamma}$ ….(6) , $\lambda_2=\frac{-(\alpha-\delta)-\sqrt{(\alpha-\delta)^2+4\gamma\beta}}{2\gamma}$….(7)

Put $\mu_1 = \alpha + \lambda_1\gamma$ ….(8) , $\mu_2 = \alpha + \lambda_2\gamma$ ….(9)

From (4), (5), (8), $y_n + \lambda_1 z_n = y_{n-1} + \lambda_1 \mu_1 z_n = \mu_1(y_{n-1} + \lambda_1 z_{n-1}) = {\mu_1}^2(y_{n-2} + \lambda_1 z_{n-2}) = \dots$

$= {\mu_1}^n(y_0 + \lambda_1 z_0)$ …. (10)

Similarly, $y_n + \lambda_2 z_n = {\mu_2}^n(y_0 + \lambda_2 z_0)$ …. (11)

If $\lambda_1 \neq \lambda_2$, we can solve (10), (11)

$$y_n = \frac{\lambda_2\mu_1^n(y_0+\lambda_1 z_0) - \lambda_1\mu_2^n(y_0+\lambda_2 z_0)}{\lambda_2-\lambda_1}, \quad z_n = \frac{\mu_2^n(y_0+\lambda_1 z_0) - \mu_1^n(y_0+\lambda_2 z_0)}{\lambda_2-\lambda_1}$$

and $x_n = \frac{y_n}{z_n} = \frac{\lambda_2\mu_1^n(y_0+\lambda_1 z_0) - \lambda_1\mu_2^n(y_0+\lambda_2 z_0)}{\mu_2^n(y_0+\lambda_1 z_0) - \mu_1^n(y_0+\lambda_2 z_0)} = \frac{\lambda_2\mu_1^n(x_0+\lambda_1) - \lambda_1\mu_2^n(x_0+\lambda_2)}{\mu_2^n(x_0+\lambda_1) - \mu_1^n(x_0+\lambda_2)}$

If $\lambda_1 = \lambda_2$, then $\mu_1 = \mu_2$ and equations (10) and (11) coincide.

To determine y_n and z_n we proceed as follows. From (10), $y_n = -\lambda_1 z_n + \mu_1^n(y_0 + \lambda_1 z_0)$ ….(12)

(12)↓(3), $z_n = \gamma[-\lambda_1 z_{n-1} + \mu_1^n(y_0 + \lambda_1 z_0)] + z_{n-1} = (\delta - \gamma\lambda_1) z_{n-1} + \gamma(y_0 + \lambda_1 z_0)\mu_1^{n-1}$ ….(13)

Put $k = \delta - \gamma\lambda_1$ $m = \gamma(y_0 + \lambda_1 z_0)$ ….(14)

Then (13) becomes $z_n = k z_{n-1} + m\mu_1^{n-1} = k(k z_{n-2} + m\mu_1^{n-2}) + m\mu_1^{n-1} = k^2 z_{n-2} + m(\mu_1^{n-1} + k\mu_1^{n-2})$

$= \ldots = k^n z_0 + m(\mu_1^{n-1} + k\mu_1^{n-2} + k^2\mu_1^{n-3} + \ldots + k^n\mu_1^0)$

$$= k^n z_0 + m\mu_1\frac{1-(k/\mu_1)^n}{1-(k/\mu_1)} = k^n z_0 + \frac{m}{\mu_1^{n-2}}\frac{\mu_1^n - k^n}{\mu_1 - k} \quad \ldots. \quad (15)$$

(15)↓(12), $$y_n = -\lambda\left[{}_1 k^n z_0 + \frac{m}{\mu_1^{n-2}}\frac{\mu_1^n - k^n}{\mu_1 - k}\right] + \mu_1^n(y_0 + \lambda_1 z_0) \quad \ldots. \quad (16)$$

From (15), (16), $x_n = \frac{y_n}{z_n}$ can be found.

(i) $x_n = \frac{x_{n-1}}{2x_{n-1}+1}$ $\Rightarrow \frac{1}{x_n} - \frac{1}{x_{n-1}} = 2$. Replace here n = 1, 2, 3, …, n and add, we have

$\frac{1}{x_n} - \frac{1}{x_0} = 2n$ $\therefore$ $x_n = \frac{x_0}{2nx_0+1}$

(ii) $x_n = \frac{x_{n-1}+1}{x_{n-1}+3}$ Put $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 3$, then

$$\lambda_1, \lambda_2 = \frac{-(\alpha-\delta) \pm \sqrt{(\alpha-\delta)^2 + 4\gamma\beta}}{2\gamma} = 1 \pm \sqrt{2}.$$

$\mu_{1,2} = \alpha + \lambda_{1,2}\gamma = 2 \pm \sqrt{2}$.

$$x_n = \frac{\lambda_2\mu_1^n(x_0+\lambda_1) - \lambda_1\mu_2^n(x_0+\lambda_2)}{\mu_2^n(x_0+\lambda_1) - \mu_1^n(x_0+\lambda_2)} = \frac{(1-\sqrt{2})(2+\sqrt{2})^n(x_0+1+\sqrt{2}) - (1+\sqrt{2})(2-\sqrt{2})^n(x_0+1-\sqrt{2})}{(2-\sqrt{2})^n(x_0+1+\sqrt{2}) - (2+\sqrt{2})^n(x_0+1-\sqrt{2})}$$

8. Put $\lambda = \frac{p}{p+q}$, $1-\lambda = 1 - \frac{p}{p+q} = \frac{q}{p+q}$, then $x_n = \lambda x_{n-1} + (1-\lambda) x_{n-2}$

$\therefore$ $x_n - x_{n-1} = \lambda x_{n-1} + (1-\lambda) x_{n-2} - x_{n-1} = (-1)(1-\lambda) x_{n-1} + (1-\lambda) x_{n-2}$

$= (-1)(1-\lambda)(x_{n-1} - x_{n-2}) = (-1)^2(1-\lambda)^2(x_{n-2} - x_{n-3}) = \ldots = (-1)^{n-1}(1-\lambda)^{n-1}(x_1 - x_0)$

$$\sum_{i=1}^{n}(x_i - x_{i-1}) = \sum_{i=1}^{n}(-1)^{i-1}(1-\lambda)^{i-1}(x_1-x_0) = (x_1-x_0)\sum_{i=1}^{n}(\lambda-1)^{i-1} = (x_1-x_0)\frac{1-(\lambda-1)^n}{2-\lambda} \quad \text{(geom. series)}$$

9. See the solution of No. 7.

10. $x_n = x_{n-1} + 2y_{n-1}\sin^2\alpha$ …. (1), $y_n = y_{n-1} + 2x_{n-1}\cos^2\alpha$ …. (2)

$x_n + \lambda y_n = (1 + 2\lambda\cos^2\alpha) x_{n-1} + (\lambda + 2\sin^2\alpha) y_{n-1}$ …. (3)

Choose λ such that $\lambda(1 + 2\lambda \cos^2 \alpha) = \lambda + 2 \sin^2 \alpha$ …. (4)

Then $2\lambda^2 \cos \alpha = 2 \sin^2 \alpha \Rightarrow \lambda = \pm\tan \alpha$ …. (5)

With these λ's, from (3) and (4) we have

$x_n + \lambda y_n = (1 + 2\lambda \cos^2 \alpha)\, x_{n-1} + \lambda(1 + 2\lambda \cos^2 \alpha)\, y_{n-1} = (1 + 2\lambda \cos^2 \alpha)(x_{n-1} + \lambda y_{n-1})$

$= (1 + 2\lambda \cos^2 \alpha)^2 (x_{n-2} + \lambda y_{n-2}) = \ldots = (1 + 2\lambda \cos^2 \alpha)^n (x_0 + \lambda y_0) = (1 + 2\lambda \cos^2 \alpha)^n \lambda \cos \alpha$ …. (6)

$\therefore$ $x_n + (\tan \alpha)\, y_n = (1 + \sin 2\alpha)^n \sin \alpha$ …. (7)

$x_n - (\tan \alpha)\, y_n = (1 - \sin 2\alpha)^n (-\sin \alpha)$ …. (8)

Solve (7) and (8) for x_n and y_n,

$$x_n = \frac{1}{2}\sin\alpha\left[(1+\sin 2\alpha)^n - (1-\sin 2\alpha)^n\right], \qquad y_n = \frac{1}{2}\cos\alpha\left[(1+\sin 2\alpha)^n + (1-\sin 2\alpha)^n\right]$$

11. $a_{n+1} - 2a_n + a_{n-1} = 1 \Rightarrow (a_{n+1} - a_n) - (a_n - a_{n-1}) = 1 \Rightarrow \sum_{i=2}^{k} (a_{i+1} - a_i) - \sum_{i=2}^{k} (a_i - a_{i-1}) = \sum_{i=2}^{k} 1$

$$\therefore \quad (a_{k+1} - a_2) - (a_k - a_1) = k - 1 \Rightarrow a_{k+1} - a_k = (k-1) + (a_2 - a_1)$$

$$\Rightarrow \sum_{k=1}^{n-1} (a_{k+1} - a_k) = \sum_{k=1}^{n-1} (k-1) + \sum_{k=1}^{n-1} (a_2 - a_1) \Rightarrow a_n - a_1 = \frac{(n-1)(n-2)}{2} + (n-1)(a_n - a_1)$$

$$\therefore \quad a_n = \frac{(n-1)(n-2)}{2} + (n-1)(a_2 - a_1) + a_1$$

12. $a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 1 \Rightarrow a_{n+2} - 3a_{n+1} + 3a_n - a_{n-1} = 1 \Rightarrow (a_{n+2} - a_{n+1}) - 2(a_{n+1} - a_n) + (a_n - a_{n-1}) = 1$

Put $u_n = a_{n+1} - a_n$, we have $u_{n+1} - 2u_n + u_{n-1} = 1$. By No. 11., we have

$$u_n = \frac{(n-1)(n-2)}{2} + (n-1)(u_2 - u_1) + u_1 \Rightarrow a_{n+1} - a_n = \frac{(n-1)(n-2)}{2} + (n-1)(a_3 - 2a_2 + a_1) + (a_2 - a_1)$$

$$\therefore \quad \sum_{k=1}^{n-1} (a_{k+1} - a_k) = \frac{1}{2}\sum_{k=1}^{n-1} (k^2 - 3k + 2) + a_3 \sum_{k=1}^{n-1} (k-1) - a_2 \sum_{k=1}^{n-1} (2k-3) + a_1 \sum_{k=1}^{n-1} (k-2)$$

$$a_n - a_1 = \frac{1}{2} \times \left[\frac{1}{6}(n-1)n(2n-1) - 3\frac{n(n-1)}{2} + 2(n-1)\right]$$

$$+ a_3 \frac{1}{2}(n-1)(n-2) - a_2\left[\frac{2(n-1)n}{2} - 3(n-1)\right] + a_1\left[\frac{n(n-1)}{2} - 2(n-1)\right]$$

$$\therefore \quad a_n = \frac{1}{6}(n-1)(n-2)(n-3) + \frac{1}{2}(n-2)(n-3)a_1 - (n-1)(n-3)a_2 + \frac{1}{2}(n-1)(n-2)a_3$$

13. If $k \neq 1$, $a_n = ka_{n-1} + l = k(ka_{n-2} + l) + l = k^2 a_{n-2} + (k+1)l = k^2(ka_{n-3} + l) + (k+1)l$

$= k^3 a_{n-3} + (k^2 + k + 1)l = \ldots = k^{n-1} + [k^{n-2} + k^{n-3} + \ldots + k + 1]\, l$

$$= k^{n-1}a_1 + \frac{k^{n-1} - 1}{k - 1}\, l$$

If $k = 1$, $a_n = a_1 + (n-1)\, l$.